Superconductivity - Assignment 5

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17 T_c upper limit in BCS

In BCS theory, the formation of Cooper pairs is mediated by phonons. There is a phonon-electron interaction quantified by the dimensionless quantity

$$\lambda := V g(\epsilon_F)$$

with V Cooper's approximate potential and $g(\epsilon_F)$ the density of states near the Fermi surface for the electrons. A thorough discussion can be found in Annett's book [1, chapter 6] and in the slides of week 6 of this course. The binding energy of the Cooper pairs (i.e. the energy gain of forming these pairs) is

$$-E = 2\hbar\omega_D e^{-1/\lambda} =: \Delta_0,$$

which is also called the gap parameter Δ_0 at zero temperature for BCS. In the weak coupling limit of the BCS theory, the case we have considered so far, it is assumed that $\lambda << 1$. It should be noted that this weak limit also means that the gap is smaller than the thermal energy of the highest excited energy phonon, which corresponds to the Debye temperature

$$\Delta < k_B\Theta_D.$$

It is when this assumption breaks down, BCS does not work and we find an upper limit to the critical temperature T_c. We will look at a way to express the critical temperature in terms we can derive, and then look at the values that maximize this critical temperature whilst still following BCS theory.

From the derivation of the BCS coherent state, this gap parameter at finite temperature is found. There is a temperature dependence $\Delta(T)$ as in figure 1. For larger temperatures, thermal energy is increased, and less energy is required to break up Cooper pairs, thus degrading the superconductivity. This puts a limit T_c.

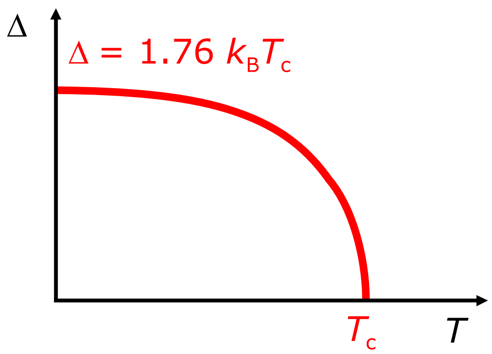


Figure 1: By taking the gap parameter to zero, we find the critical temperature. Figure from the slides of lecture 7.

We will mostly follow the derivation by Waldram [2, paragraph 7.9, mostly p.128–130]. The superconducting state breaks down at high temperature, at which also Δ vanishes so that the gap parameter is a good order parameter for the state.

Let's consider the gap parameter

$$\Delta_{\vec{k}} = -\sum_{\vec{k'}} (1 - 2f_{\vec{k'}}) u_{\vec{k'}} v_{\vec{k}} V_{\vec{k'}\vec{k}},$$

with u and v occupation functions for the BCS state, f the Fermi occupation number, and V the potential between the states. Minimizing $\Delta_{\vec{k}}$ and taking that $V_{\vec{k'}\vec{k}} = -V$ is constant gives us a self-consistent relation for the gap parameter. We also recognize that the states that we sum over all all those states such that they have smaller energy than the highest excited phonon.

$$\Delta_{\vec{k}} = V \sum_{\epsilon_{\vec{k'}}} (1 - 2f_{\vec{k'}}) \frac{\Delta_{\vec{k'}}}{2E_{\vec{k'}}}.$$

Now the right-hand side is independent of $\vec{k}$ but does contain $\Delta_{\vec{k'}}$. We can thus conclude that the gap parameter should be constant over all states $\vec{k}$! That means we can divide both sides by it, giving us

$$1 = V \sum_{\epsilon_{\vec{k'}}} (1 - 2f_{\vec{k'}}) \frac{1}{2E_{\vec{k'}}}.$$

Converting the equation to an integral, and substituting in $f(E) = [\exp{(E/(k_B T))} + 1]^{-1}$ and $E = \sqrt{\epsilon^2 + \Delta(T)^2}$ yields

$$1 = 2g(\epsilon_F)V \int_0^{k_B \Theta_D} \frac{1 - 2[\exp{(E/(k_B T))} + 1]^{-1}}{2\sqrt{\epsilon^2 + \Delta(T)^2}} \mathrm{d}\epsilon.$$

I believe Waldram that one could find that

$$T_c = 1.14\Theta_D \exp{(-1/(g(\epsilon_F)V))} = 1.14\Theta_D \exp{(-1/(\lambda)}$$

from this nice equation.

As limiting value, we take $\lambda = 0.3$, as was posed as a reasonable limit for the weak coupling by Alix in lecture 7, although Waldram [2] thinks it is more like $\lambda \approx 0.4$.

For metals, Waldram thinks $\Theta_D \leq 300\,\mathrm{K}$ is a good limit. This leads to our final maximum

$$T_c \leq 1.14 \cdot 300 \cdot \exp{(-1/0.4)} \approx 28\,\mathrm{K}.$$

(Using $\lambda = 0.3$ yields an even lower $T_c \leq 12\,\mathrm{K}$.)

For larger T_c values, larger binding of Cooper pairs would be needed to overcome the thermal energy. This means our assumption of weak coupling breaks down, making most of the derivation invalid without further arguments.

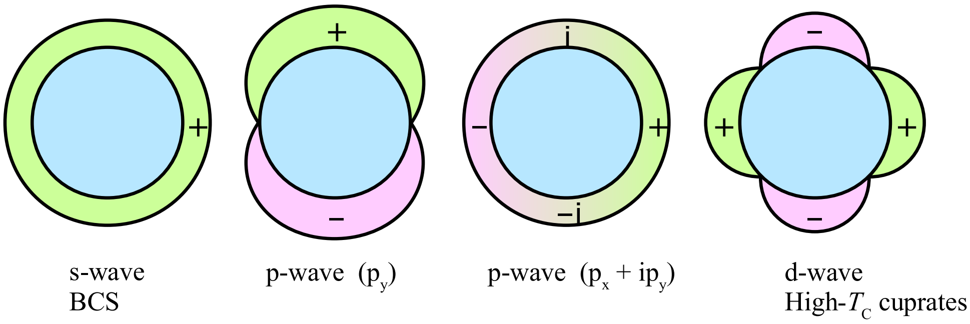


Figure 2: Different types of superconductor waves have different node patterns. The figure is from the slides of lecture 9 by Alix McCollam.

18 Penetration depth λ and measuring it

There are many species of superconductors. Conventional superconductors we can describe using BCS theory or some extension of it. Others we do not yet have a theory for. Some are type-I, others type-II. What they do have in common, is that they can be characterized by some key quantities. Starting with macroscopic ones, we have the critical temperature T_c and critical field(s) H_c. The microscopic behavior is described by three characteristic lengths[1, p.62]: the coherence length ξ of the Cooper pairs, the penetration depth λ of the external field, and the mean free path ℓ of the electrons. These quantities are related to the energy band gap around the Fermi surface in BCS theory. A nice table summarizing these quantities can be found in [2, table 10.1, p. 191]. In this essay, we will take a look at what the penetration depth can tell us about the superconducting energy gap, and will go into measuring the penetration depth.

The band gap energy $\Delta(k)$ is a useful order parameter for superconductivity. It can tell us a lot about what kind of superconductor we are dealing with. It is not the maximum value of Δ we are after, but its variation in momentum space, and more specifically any nodes in it. Results on the relation between the nodes of the energy gap and the type of superconducting wave we deal with can be seen in figure 2. The regions with opposite sign correspond to regions of repulsion, whereas same sign regions have attraction. Now it is our job to connect this to λ.

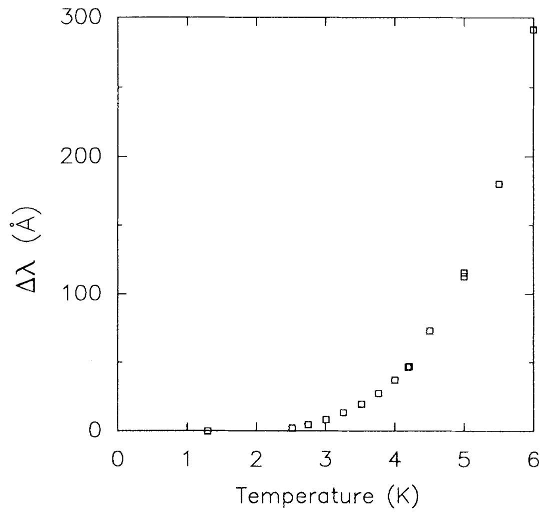
In the theory by the London theory of superconductivity, the penetration depth is related to the superfluid density n_s[1, ch. 3, ch. 7.5] (of the superfluid model) as

$$\lambda_L(T) = \sqrt{\frac{m_e^*}{\mu_0 e^2 n_s(T)}}.$$

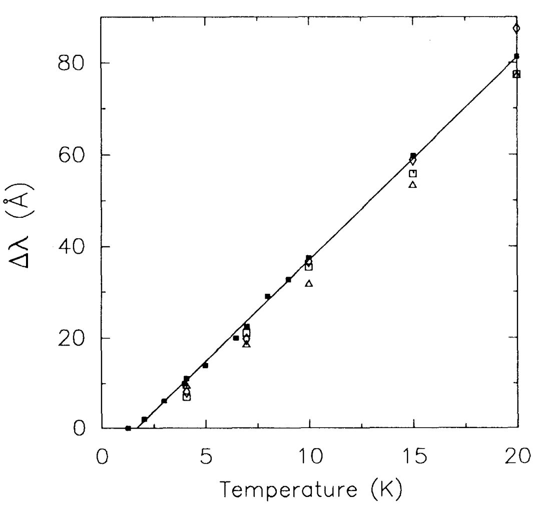
If $n_s(T)$ can be related to energy gap Δ, so can λ, and luckily we can. If there is a node in $\Delta(k)$ for some k, it means that there will be states available for any energy we put in. This in turn implies a linearly increasing relation to $\lambda(T) = \lambda(0) + cT$ for some constant c.

In the weak coupling limit of BCS theory, around the Fermi sphere, we see a constant band gap. There thus are no nodes. BCS describes s-wave superconductors. For other types, this is not the case: there is gap anisotropy. A result like in figure 2 can thus tell us what kind of superconductor we see. Looking at $\lambda(T)$, we find plots as in figure 3.

But now the question is how we can measure this gap anisotropy in practice. To image the complete k-dependence of the gap, it is required that the probe is sensitive to the direction of the electron momenta[2, p.207], for which there are multiple methods. A direct way would be to use ARPES, as that directly probes the band gap energy and is angular resolved, thus yielding a k-dependent measurement. However, we want to take a look at a different approach. We will focus on using $\lambda(T)$ measurements using tunnel diode oscillators (TDO)[4], as that technique is used in the provided paper, and we just discussed the relation between λ and the band gap. Do note that angular information will not be obtained this way.



(a) For superconductors without nodes (s-wave, BCS), there is a constant gap energy, resulting in $\lambda(T) \propto [n_s(0)(1 - \alpha \exp(\frac{\Delta}{k_B T}))]^{-1/2}$.



(b) For superconductors with line nodes, such as d-wave and some p-wave, $\lambda(T) \propto T$ is observed as was expected.

Figure 3: Both figures are from [3].

A thorough discussion about λ measurements using a TDO is presented in [5]. The idea is to measure the resonant frequency of an LC-circuit which inductance L changes as function of the penetration depth. A piece of superconductor material is inserted in the coil of the LC-circuit, preferably a slab, cylinder or sphere, as these yield exact results to the London equations that are used for determining the dependence. The LC-circuit is turned on by some AC signal. This in turn induces an alternating magnetic field H inside the coil. Following the London equations, this induces a magnetic moment m inside the superconductor sample that is linear to the field and depends on the geometry of the sample, thus $m = C(\lambda)H$. This magnetic moment in its turn affects the inductance of the coil, resulting in a resonant frequency change

$$\delta f = f(\text{with sample}) - f(\text{without sample}) = Gm = GC(\lambda)H,$$

with G the effective volume of the coil. As determining the geometry and field directions for C is quite error prone and hard due to the smallness of the quantities, they are usually not determined. They are, however, kept constant, and λ is what is varied by changing the temperature such that we can easily write

$$\Delta\lambda = \lambda(T) - \lambda(0).$$

With knowledge about $\lambda(0)$ from other sources, $\lambda(T)$ is determined by determining $\Delta\lambda$ from δf. Now the superfluid density can be determined.

In [4], heavy-fermion superconductor $CeCoIn_5$ is investigated using the TDO technique to measure its penetration depth. It is an unconventional superconductor, and the question is what type of wave-symmetry it exhibits. The paper found a non-linear $\lambda(T)$ relation. See figure 4 for their results. They do conclude that the material is in a $d_{x^2-y^2}$ superconductor ground state. I would expect there to be no nodes in the band gap energy, in this case, which however is the case. The authors also seem puzzled at the beginning. They suspect strong-scattering impurities to alter the $\lambda(T)$ relation. To exclude this possibility, they checked a couple of possible explanations. Purity was checked and impurity content was determined to be a factor 100 smaller than the deviation in $\lambda(T)$ would imply. Other theories were also ruled out, on impossibility of far-fetchedness. They conclude by proposing non-Fermi-liquid renormalisation in both the normal and superconducting state of $CeCoIn_5$ to take place, yielding the well fitting relation as seen in the inset of figure 4. This means that their would be quantum criticality in

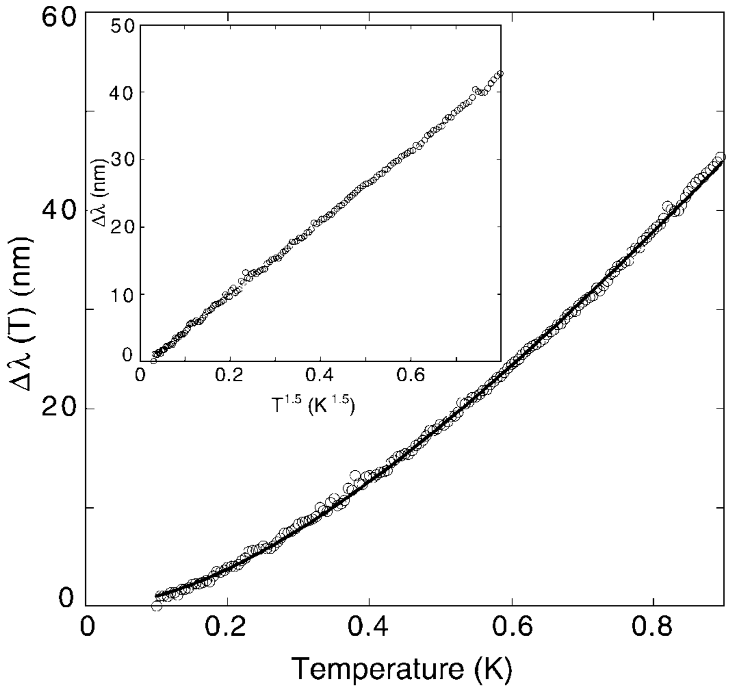


Figure 4: For $CeCoIn_5$, $\lambda(T) \propto T^2/(T - T^*)$ is plotted in the main plot. In the inset, the concluding hypothesis of the authors [4] is presented, i.e. $\lambda(T) \propto T^{1.5}$.

the superconducting state, i.e. a phase transition at zero temperature. That would be exotic. In conclusion, the behavior of $CeCoIn_5$ was not explained with certainty at the point this paper was published (2003), although quantum criticality was a possibility. However, years later (2014), further research supports their hypothesis [6].

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