Onderwerp:

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Superconductivity assignments 4 Kees van Kempen 1/8

13a

$$\langle\alpha|\beta\rangle = e^{-\frac{|\alpha|^2+|\beta|^2}{2}}\left(\langle\psi_0^*| + \frac{\alpha^*}{\sqrt{1!}}\langle\psi_1^*| + \frac{\alpha^2}{\sqrt{2!}}\langle\psi_2^*| + \frac{\alpha^3}{\sqrt{3!}}\langle\psi_3^*| + \dots\right)$$ ← no *

$$\left(|\psi_0\rangle + \frac{\beta}{\sqrt{1!}}|\psi_1\rangle + \frac{\beta^2}{\sqrt{2!}}|\psi_2\rangle + \frac{\alpha^3}{\sqrt{3!}}|\psi_3\rangle + \dots\right)$$

$$\cancel{|\langle\alpha|\beta\rangle|^2 = \langle\alpha|\beta\rangle\cdot\langle\beta|\alpha\rangle}$$

$$= e^{-\frac{|\alpha|^2+|\beta|^2}{2}}\cdot\left(\langle\psi_0^*|\psi_0\rangle + \frac{\alpha^*\beta}{\sqrt{1!}^2}\langle\psi_1^*|\psi_1\rangle + \frac{\alpha^{*2}\beta^2}{\sqrt{2!}^2}\langle\psi_2^*|\psi_2\rangle + \frac{\alpha^{*3}\beta^3}{\sqrt{3!}^2}\langle\psi_3^*|\psi_3\rangle + \dots\right)$$

as $\langle\psi_i|\psi_j\rangle = \delta_{ij}$, thus other terms disappear

$$= e^{-\frac{|\alpha|^2+|\beta|^2}{2}}\cdot\sum_{n=0}^{\infty}\frac{(\alpha^*\beta)^n}{n!} = e^{-\frac{|\alpha|^2+|\beta|^2}{2}}\cdot e^{\alpha^*\beta}$$

Thus $|\langle\alpha|\beta\rangle|^2 = \langle\alpha|\beta\rangle\cdot\langle\alpha|\beta\rangle^*$

$$= e^{-(|\alpha|^2+|\beta|^2)}\cdot e^{\alpha^*\beta}\cdot e^{\alpha\beta^*}$$

$$= e^{-(|\alpha|^2+|\beta|^2)+\alpha^*\beta+\alpha\beta^*}$$

$$= e^{-(\alpha\alpha^*+\beta\beta^*)+\alpha^*\beta+\alpha\beta^*}$$

$$= e^{-(\alpha-\beta)(\alpha^*-\beta^*)} = e^{-(\alpha-\beta)(\alpha-\beta)^*} = e^{-|\alpha-\beta|^2}$$

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23/b

1) b Let $|\psi\rangle = \sum_{n=0}^{\infty} c_n |\psi_n\rangle$ with $c_n \in \mathbb{C}$ and ψ_n basis vectors

As such, applying the annihilation operator $\hat{c}$ to it yields

$$\hat{c}|\psi\rangle = \hat{c} \sum_{n=0}^{\infty} c_n |\psi_n\rangle = \sum_{n=0}^{\infty} c_n \hat{c}|\psi_n\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n} |\psi_{n-1}\rangle + \underbrace{c_0 \cdot 0 |\psi_{-1}\rangle}_{=0}$$

$$= \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1} |\psi_n\rangle$$

Which indeed is an eigenstate to $\hat{c}$ if $\exists a \in \mathbb{C}$ such that $\forall n \in \mathbb{N}$ $c_n = a \cdot c_{n+1} \sqrt{n+1}$, ~~which is possible~~.

$$\Leftrightarrow \frac{c_{n+1}}{c_n} = \frac{1}{a\sqrt{n+1}}.$$

It follows then that $a = \frac{1}{\alpha}$ in the form of coherent states in 1.3 a for $|\alpha\rangle$.

(Yes, this was not asked but felt cool to look into.)

Now we apply the creation operator $\hat{c}^{\dagger}$ to $|\psi\rangle$.

$$\hat{c}^{\dagger}|\psi\rangle = \sum_{n=0}^{\infty} \hat{c}^{\dagger} c_n |\psi_n\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n+1} |\psi_{n+1}\rangle$$

$$= \sum_{n=1}^{\infty} c_{n-1} \sqrt{n} |\psi_n\rangle$$

For $|\psi\rangle$ to be an eigenstate of $\hat{c}^{\dagger}$ we thus require $\exists b \in \mathbb{C}$ such that $\forall n \in \mathbb{N}$

$$c_n = b \cdot c_{n+1} \sqrt{n} \quad \Leftrightarrow c_{n+1} = \frac{1}{b\sqrt{n}} c_n$$

and $c_0 = 0$ as the basis vector $|\psi_0\rangle$ drops out under application of $\hat{c}^{\dagger}$. By induction, then, if $c_0 = 0$, $c_n = 0$ $\forall n \in \mathbb{N}$, and $|\psi\rangle \equiv 0$, so there exists no eigenstate to $\hat{c}^{\dagger}$.

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14 For the Cooper problem we assume a two electron system with wave-functions of position $\vec{r}_i$ and spins σ_i

$$\psi(\vec{r}_1, \sigma_1, \vec{r}_2, \sigma_2) = e^{i\vec{K}\cdot\vec{R}} \varphi(\vec{r})\, \varphi^{spin}(\sigma_1, \sigma_2)$$

with $\vec{K}$ centre of mass wavenumber and $\vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$ centre of mass position. We take this form as

We assume the potential to only depend on the relative position $\vec{r} = \vec{r}_1 - \vec{r}_2$. Furthermore, the lowest energy state corresponds to $\vec{K}=0$ (See Annett, p. 134). We will look at the SE in the relative position $\vec{r}$.

$$\left[\frac{-\hbar^2 \nabla_{\vec{r}}^2}{2\cdot(\frac{1}{2}m)} + V(\vec{r}_1 - \vec{r}_2)\right]\varphi(\vec{r}) = \left[\frac{-\hbar^2 \nabla_{\vec{r}}^2}{m} + V(\vec{r})\right]\varphi(\vec{r}) = E\varphi(\vec{r})$$

reduced mass

By taking the Fourier transform for this relative motion, we write

$$\varphi(\vec{r}) = \sum_{|\vec{k}| \in [k_F,\, k_F + \frac{\omega_D}{v}]} \varphi_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}}$$

v is the Bloch wave group velocity at the FS.

as we only care about state in the shell above the Fermi surface until the Debye frequency above it. I will omit these restraints in writing.

Putting this into the SE yields the following

$$\left[\frac{-\hbar^2 \nabla_{\vec{r}}^2}{m} + V(\vec{r})\right] \sum_{\vec{k}} \varphi_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}} = E \sum_{\vec{k}} \varphi_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}}$$

$$\Leftrightarrow \frac{+\hbar^2}{m}\sum_{\vec{k}} \varphi_{\vec{k}}\, k^2 e^{i\vec{k}\cdot\vec{r}} + V(\vec{r})\sum_{\vec{k}} \varphi_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}} = E\sum_{\vec{k}} \varphi_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}}$$

with $k = |\vec{k}|$. ~~We we substitute~~

From this point, I keep writing $\varphi(\vec{k})$ for $\varphi_{\vec{k}}$. Note it's the Fourier transform of $\varphi(\vec{r})$!

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Now we will ~~back~~ transform ~~back~~ to $\varphi_{\vec k}$, which means taking ~~the k~~ one $\vec{k}$ from the $\frac{\hbar^2}{m}$ and E term, and integrating over all space for the potential term.

$$\Rightarrow \frac{\hbar^2 k^2}{m}\varphi_{\vec k} + \int d\vec r\, V(\vec r)\underbrace{\varphi(\vec r)}_{=\sum_{k}\varphi_{\vec k} e^{i\vec k\vec r}} e^{-i\vec k\cdot\vec r} = E\varphi_{\vec k}$$

~~for the Fourier transform~~

$$(\Leftrightarrow) \int d\vec r\, V(\vec r)\varphi(\vec r) e^{-i\vec k\cdot\vec r} = \left(E - \frac{\hbar^2 k^2}{m}\right)\varphi_{\vec k} \qquad \text{I}$$

After transforming $V(\vec r)\to V(\vec q) = V(\vec k_1 - \vec k_2)$, the integral can be solved by using the convolution thm (i.e. for g, h and their Fourier transforms G, H, the Fourier transform of $g*h$ ~~equals~~ equals ~~G*H~~ G H)

$$\int d\vec r\, V(\vec r)\varphi(\vec r) e^{-i\vec k\vec r} = \cancel{\int d\vec r\, V(\vec r) \int d\vec r\, \varphi(\vec r)}$$

~~due to Fourier transform~~

$$\vec q = \vec k_1 - \vec k_2$$

$$= \iint \underbrace{\left(\frac{1}{2\pi}\right)^{3n} e^{i\vec q\cdot\vec r}}_{\text{due to Fourier transform in n dimensions}} V(\vec q)\, d\vec q \cdot d\vec r\, \varphi(\vec r)\, e^{-i\vec k\cdot\vec r}$$

↑ ~~ranging in shell above~~ k_F ranging from 0 to $\frac{\omega_D}{v}$, such that $\vec k'$ is in the shell

$$= \iint \left(\frac{1}{2\pi}\right)^{3n} V(\vec q)\varphi(\vec r)\, e^{-i(\vec k-\vec q)\cdot\vec r}\, d\vec q\, d\vec r$$

Now the convolution thm:

$$= \int \left(\frac{1}{2\pi}\right)^{3n} V(\vec k - \vec k')\varphi(\vec k')\, d\vec k' = \left(E - \frac{\hbar^2k^2}{m}\right)\varphi_{\vec k}$$

* ↑ over shell

↑ back to I

(I am a little sloppy in using integrals ~~and discrete sums~~ and sums for $\vec k$ but it does kinda the same.)

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As $V(\vec{r})$ is quite spread out in real space, ~~and~~ $V(\vec{k})$ is strongly peaked. We thus take $V(\vec{k})$ to be constant over our range of interest, i.e. from the Fermi surface to $\frac{\omega_D}{2}$ above it, setting $V(\vec{k}) = -V_0$ in that range, otherwise zero.

$$\Rightarrow \int -\left(\frac{1}{2\pi}\right)^{3n} V_0 \, \varphi(\vec{k}') \, d\vec{k}' = -\left(\frac{1}{2\pi}\right)^{3n} V_0 \int \varphi(\vec{k}') \, d\vec{k}' = -\left(\frac{1}{2\pi}\right)^{3n} V_0 \cdot C$$

Shell above k_F

for which I used self-consistency in the integral over $\varphi(\vec{k}')_n$ with constant C.

$$\Rightarrow -\left(\frac{1}{2\pi}\right)^{3n} V_0 C = \left(E - \frac{\hbar^2 k^2}{m}\right) \varphi(\vec{k})$$

$$\Leftrightarrow \varphi(\vec{k}) = -V_0 C \left(\frac{1}{2\pi}\right)^{3n} \frac{1}{E - 2\varepsilon_{k}} \qquad \text{with } \varepsilon_k := \frac{\hbar^2 k^2}{2m}$$

Again, self-consistency can be used:

$$\int \varphi(\vec{k}') \, d\vec{k}' = C = \int -V_0 C \left(\frac{1}{2\pi}\right)^{3n} \frac{1}{E - 2\varepsilon_{k'}} \, d\vec{k}'$$

$$\Leftrightarrow 1 = -\frac{V_0}{(2\pi)^{3n}} \int \frac{d\vec{k}'}{E - \frac{\hbar^2 k'^2}{m}}$$

~~Wolfram Alpha~~

Now I want to evaluate the integral in different dimensions n. ~~I use 3D in the transformations above but can trivially switch to D dimensions as until now it only affected the power of $\frac{1}{2\pi}$.~~

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So,

$$I = \frac{-V_0}{(2\pi)^d}\int \frac{d\vec{k}'}{E - \frac{\hbar^2 k'^2}{m}}$$

In 1D, from $k' = 0$ to k_F,

$$I = -\frac{V_0}{2\pi}\int_{k'=0}^{k_F} \frac{dk'}{E - \frac{\hbar^2 k'^2}{m}} = -\frac{V_0}{2\pi}\frac{\sqrt{m}}{\sqrt{|E|}\hbar}\arctan\left(\frac{\hbar k'}{\sqrt{|E| m}}\right)\Bigg|_{k'=0}^{k_F}$$

~~use~~

$$= -\frac{V_0}{2\pi}\frac{\sqrt{m}}{\sqrt{|E|}\hbar}\arctan\left(\frac{\hbar k_F}{\sqrt{|E| m}}\right)$$

$$= -\frac{V_0}{2\pi}\frac{\sqrt{m}}{\sqrt{|E|}\hbar}\arctan\left(\frac{2\sqrt{\varepsilon_{k_F}}}{\sqrt{|E|}}\right)$$

For $\sqrt{\varepsilon_{k_F}} \ll \sqrt{|E|}$:

$$\simeq -\frac{V_0}{2\pi}\frac{\sqrt{m}}{\sqrt{|E|}\hbar}\frac{2\sqrt{\varepsilon_{k_F}}}{\sqrt{|E|}} = -\frac{V_0}{\pi}\frac{k_F}{|E|}$$

Allbus negative E.

For 2D and 3D it seems kinda the same. If ~~we~~ you use ~~the~~ polar and spherical coordinates, only prefactors change.

For the thin shell, only limits change.

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15 Using the SE ~~from~~ above with wave function of form

$\psi(\vec{r}_1, \sigma_1, \vec{r}_2, \sigma_2) = \varphi(\vec{r})\, \varphi^{spin}(\sigma_1, \sigma_2)\, \varphi_R(\vec{R})$

~~We can~~ ~~write~~ which corresponds to $\varphi_R(\vec{R})$ in

$\frac{\hbar^2 \nabla_R^2}{2\cdot(2m)} \varphi_R(\vec{R}) = E_R\, \varphi_R(\vec{R})$ with solution of form $\varphi_R(\vec{R}) = e^{i\vec{k}\cdot\vec{R}}$

E_R is indeed minimal for $\vec{K} = 0$.

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1i $|\Psi\rangle = \sum_{k} \alpha_k \hat{c}^\dagger_{-\vec{k},-\sigma} \hat{c}_{\vec{k},\sigma} |FS\rangle$

$\hat{H}|\Psi\rangle = E|\Psi\rangle$

$$\hat{H} = \sum_{k,\sigma} \varepsilon_k \hat{c}^\dagger_{k,\sigma} \hat{c}_{k\sigma} + \frac{1}{2} \sum_{q,k_1,\sigma_1,k_2,\sigma_2} V_{k_2,q} \hat{c}^\dagger_{k_1-q} \hat{c}^\dagger_{k_2+q} \hat{c}_{k_2} \hat{c}_{k_1}$$

1b a Show: $E g_{k'} = E_{FS} g_{k'} + 2\varepsilon_{k'} g_{k'} + \sum_{k} V_{k,k'-k} \alpha_k$

Note $\alpha_k \equiv g_k$

$$\hat{H}|\Psi\rangle = \left[\sum_{k,\sigma} \varepsilon_k \hat{c}^\dagger_{k,\sigma} \hat{c}_{k\sigma} + \frac{1}{2} \sum_{q,k_1,\sigma_1,k_2,\sigma_2} V_{k_2,q} \hat{c}^\dagger_{k_1-q} \hat{c}^\dagger_{k_2+q} \hat{c}_{k_2} \hat{c}_{k_1}\right] \sum_{k} \alpha_k \hat{c}^\dagger_{-\vec{k},-\sigma} \hat{c}^\dagger_{k,\sigma} |FS\rangle.$$

Let's apply $\langle k',\sigma'| = \langle FS| \hat{c}^\dagger_{-\vec{k}',-\sigma'} \hat{c}^\dagger_{\vec{k}',\sigma}$ from the left.

$$\langle k',\sigma'|E|\Psi\rangle = E g_{k'} = \langle FS| \hat{c}^\dagger_{-k',\sigma'} \hat{c}_{\vec{k}',\sigma}\left[\sum_{k,\sigma} \varepsilon_k \hat{c}^\dagger_{k,\sigma} \hat{c}_{k,\sigma} + \frac{1}{2} \sum_{\substack{q,k_1,\sigma_1\\k_2,\sigma_2}} V_{k_2,q} \hat{c}^\dagger_{k_1-q} \hat{c}^\dagger_{k_2+q} \hat{c}_{k_2} \hat{c}_{k_1}\right] \sum_{k} \alpha_k \hat{c}^\dagger_{-\vec{k},-\sigma} \hat{c}^\dagger_{\vec{k},\sigma} |FS\rangle$$

Splitting this eqn into parts by the terms of $\hat{H}$ helps a lot.

① $\langle FS| \hat{c}^\dagger_{-\vec{k}',-\sigma'} \hat{c}^\dagger_{\vec{k},\sigma} \sum_{k\sigma} \varepsilon_k \hat{c}^\dagger_{k\sigma} \hat{c}_{k\sigma} \cdot \sum_{k} \alpha_k \hat{c}^\dagger_{-\vec{k},-\sigma} \hat{c}^\dagger_{\vec{k},\sigma} |FS\rangle$.

If $k \neq k'$, the $\hat{c}$ and $\hat{c}^\dagger$ will commute!

Only ~~if~~ if in the second sum $(\vec{k},\sigma) = \pm(\vec{k}',\sigma)$, the result will yield $\varepsilon_{k'} \alpha_{k'}$, thus two times, for $\hat{c}_{k,\sigma} \hat{c}^\dagger_{\vec{k},\sigma} |FS\rangle$ ~~n~~ $|FS\rangle$, and for each pair in the second and last sum, part of $E_{FS} g_{k'}$ is contributed

② The potential term only yields $V_{k_2,q}$ if $q = k'-k$:

$\langle FS| \hat{c}^\dagger_{k}$ the last sum creates pair $(-\vec{k},-\sigma) + (\vec{k},\sigma)$, the middle V only acts for $k_1 - q$ or $k_2 + q$ equal to $\pm k$.

Enough.