Superconductivity - Assignment 3

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8 Nb_3Sn cylinder

Consider a cylinder of Nb_3Sb. From lecture 4, we have the following properties for Nb_3Sn: $T_c = 18.2\,\mathrm{K}$, $\xi = 3.6\,\mathrm{nm}$, $\lambda = 124\,\mathrm{nm}$, $\kappa = \frac{\lambda}{\xi} = 34 > \frac{1}{\sqrt{2}}$, which means we are indeed dealing with a type-II superconductor. As $B_{c1} < B_E < B_{c2}$, the cylinder is in the vortex state. From the previous set of assignments, we know what the currents in the cylinder look like. From free energy considerations, we have found in lecture 4 that for type-II superconductors, it is favorable to allow flux quanta inside the superconductor in this vortex state. In this derivation, the contribution of one flux quantum is considered, but the consideration holds for many vortices, until they start to interact and repel eachother. At that point, the vortex-vortex interaction orders the vortices in a lattice. When the vortex cores start to overlap, there are no superconducting regions left, thus the material enters the normal conducting state.[1] Minimizing the free energy over the flux shows the energy is lowered for determined thresholds $B_{c1} < B_E < B_{c2}$.

Let's start with the result from said free energy considerations. The average field inside the cylinder is given by the following self-consisting equation as

$$B = B_E - \frac{\phi_0}{8\pi\lambda^2} \ln \frac{\phi_0}{4\exp(1)\xi^2 B}.$$

Plugging in the values for Nb_3Sn, $B_E = 1\,\mathrm{T}$, and $\phi_0 = 2.0678\,\mathrm{Wb}$, B is found as $B = 0.986\,\mathrm{T} \approx B_E$ by intersection. This is in the range as provided in the assignment ($B = (0.981 \pm 0.019)\,\mathrm{T}$).

To investigate the inhomogenity of the field inside the cylinder, we look at the gradient ∇B inside the material. As we assume a vortex lattice that fully fills the cross section of the cylinder, and we assume that the fields due to each vortex die out quickly enough to not overlap, it suffices to calculate the gradient over just one vortex. These assumptions coincide with slide 15 of lecture 4, from which I took figure 8.

[1] I wanted to paint a complete picture althought it is not needed to answer the question.

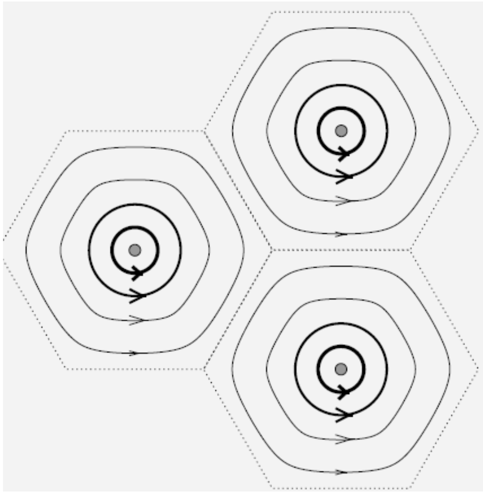

Figure 1: The vortices are arranged in a lattice to maximize their distance, as this lowers their repulsive interaction and thus the energy.

On slide 19 from the same week, we find an expression $B(r)$ for the field at distance r from the vortex core as

$$B = \frac{\phi_0}{2\pi\lambda^2} K_0(r/\lambda) = B_0 K_0(r/\lambda),$$

where K_0 is the modified Bessel function of the second kind. For small r (i.e. $r << \lambda$), we can approximate this and find that

$$K_0 \propto -\ln(r/\lambda),$$

and notice a singularity at $r = 0$. For the gradient we thus find

$$\nabla B \propto \frac{\partial K_0}{\partial r}(r/\lambda) \propto \frac{\partial - \ln(r/\lambda)}{\partial r} = -\lambda/r.$$

The size of the supercurrent density has the same relation, $J_S \propto 1/r$.

9 Superconducting wire

(a) The voltage $U = 1.5 \times 10^{-5}$ V across the wire of length $\ell = 0.08$ m induces a current J_t. Due to the presence of the magnetic field $B = 5$ T, if the vortices move with velocity v_L, a Lorentz force f_L per vortex acts on the vortices. This results in a power input $P_L = f_L v_L = J_t B v_L$ per vortex. This power should come from the current induced by the voltage, thus $P_L = \epsilon J_t = \frac{U}{\ell} J_t$. Equating these expressions and rewriting yields

$$v_L = \frac{U}{B\ell} = 3.75 \times 10^5 \,\mathrm{m\,s^{-1}}.$$

(b) The vortices are aranged in a lattice with separation $r_{sep} = \sqrt{\frac{\Phi_0}{B}}$. They move along the wire with velocity v_L as determined above. The expected frequency is then given by their velocity over the separation, as that is the period of the changing fields due to the vortices:

$$f = \frac{v_L}{r_{sep}} = \frac{U}{B\ell}\sqrt{\frac{B}{\Phi_0}} = \frac{U}{\ell\sqrt{B\Phi_0}} = 1.84\,\mathrm{kHz},$$

where we used that $\Phi_0 = 2.067 \times 10^{-15}$ V s. This is very close to what is written in the assignment, but not precisely the same, so maybe I used a different value for Φ_0.

10 Fine type-II superconducting wire

11 Critical currents

(a) Silsbee's rule states that the supercurrents through the wire must not generate magnetic fields in excess of B_c at the surface of the wire. We assume that the supercurrent is maximal at the surface with a maximum value of J_{max}, and that the supercurrent decays linearly from the surface to zero at a penetration depth λ deep. We thus find a relation for the supercurrent as function of the cylindrical radius r as

$$J_s(r) = \frac{J_{max}}{\lambda}\left[r - R + \lambda\right].$$

Now we can use the Maxwell-Ampère law to find this value for J_{max}.

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_r \mu_0 \iint \vec{J_s} \cdot d\vec{S}$$

Using $\vec{B} = \vec{B_c}$, $\mu_r = 1$ as we're calculating the field outside the sc, $J_s = J_s(r)$, the area over which $d\vec{S}$ runs to be the small ring from $r = R - \lambda$ to $r = R$, and the path along which $d\vec{\ell}$ runs to be the loop $2\pi R$ along the surface of the wire. This gives us

$$2\pi R B_c = \mu_0 \int_{\phi=0}^{2\pi} \int_{r=R-\lambda}^{R} \frac{J_{max}}{\lambda}\left[r - R + \lambda\right] r dr d\phi.$$

Solving the integral over ϕ results in

$$R B_c = \mu_0 \int_{r=R-\lambda}^{R} \frac{J_{max}}{\lambda}\left[r - R + \lambda\right] r dr = \frac{\mu_0 J_{max}}{\lambda}\left[\frac{r^3}{3} - \frac{(R+\lambda)r^2}{2}\right]_{r=R-\lambda}^{R}.$$

Solving for J_{max}, this yields the beautiful expression

$$J_{max} = \frac{6 B_c \lambda R}{\mu\left[4\lambda^3 - 9\lambda^2 R + 3\lambda R^2 - 3\lambda R + 3R^3 - 3R^2\right]}.$$

(b)

12 A weak junction

See the code in appendix A.

It unfortunately does not seem to produce any useful results. In the code, I left many comments as it is mostly in a debugging state.

References

A Program to task 12

```
#!/bin/env python3

import numpy as np
from scipy.integrate import odeint
from matplotlib import pyplot as plt
import pandas as pd
```

```
def phi_dot(phi, t, I_DC, I_RF):
    R        = 10.e-3 #Ohm
    I_J      = 1.e-3 #A
    omega_RF = 2*np.pi*.96e9 #rad/s
    hbar     = 1.0545718e-34 #m^2kg/s
    e        = 1.60217662e-19 #C

    return 2*e*R/hbar*( I_DC + I_RF*np.cos(omega_RF*t) - I_J*(np.sin
        (phi)) )
    # I attempted to solve it without the constants, as I suspected
        overflows
    # were occurring. The next line did not improve the result.
    #return R*( I_DC + I_RF*np.sin(omega_RF*t) - I_J*(np.sin(phi)) )

# We need an initial value to phi
phi_0 = 0

# Let's try it for a lot of periods
N_points = 10000
t = np.linspace(0, 100, N_points)

hbar     = 1.0545718e-34 #m^2kg/s
e        = 1.60217662e-19 #C

df = pd.DataFrame(columns=['I_DC', 'I_RF', 'V_DC_bar'])

# For testing:
#phi = odeint(phi_dot, phi_0, t, (.5e-3, .5e-3))[:, 0]
#for I_DC in [1e-4, .5e-3, 1.e-3, 1.5e-3, 2.e-3, 2.5e-3]:

for I_DC in np.arange(0, 1e-3, 1e-5):
    for I_RF in [0., .5e-3, 2.e-3]:
        # The individual solutions for phi do seem sane, at least,
            the ones
        # I inspected.
        phi = odeint(phi_dot, phi_0, t, (I_DC, I_RF))
        # I initially thought to average over the tail to look at
            the asymptotic behaviour.
        #N_asymp = N_points//2
        #V_DC_bar = hbar/(2*e)*np.mean(phi[N_asymp:]/t[N_asymp:])
        # Then I choose to just take the last point to see if that
            gave better results.
        V_DC_bar = hbar/(2*e)*phi[-1]/t[-1]
        print("For I_DC =", I_DC, "\t I_RF = ", I_RF, "\twe find
            V_DC_bar =", V_DC_bar)
        df = df.append({'I_DC': I_DC, 'I_RF': I_RF, 'V_DC_bar':
            V_DC_bar}, ignore_index = True)

## Plotting the thing
plt.figure()
```

```
plt.xlabel("$\\overline{V_{DC}}$")
plt.ylabel("$I_{DC}$")

for I_RF in df.I_RF.unique():
    x, y = df[df.I_RF == I_RF][["V_DC_bar", "I_DC"]].to_numpy().T
    plt.plot(x, y, label="$I_{RF} = " + str(I_RF) + "$")

plt.legend()
plt.show()
```