Superconductivity - Assignment 2

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4 Temperature dependence in Landau model

In the Landau model, free energy is given as function of order parameter ψ and temperature T as

$$\mathcal{F} = a(T - T_c)\psi^2 + \frac{\beta}{2}\psi^4.$$

The equilibrium state as function of temperature T is the state of minimal free energy with respect to the order parameter $\psi(T)$. This point we call $F_0(T)$ with order parameter $\psi_0(T)$. For this, we will take the derivative of F with respect to ψ and equate it to zero.

$$0 = \frac{\partial \mathcal{F}}{\partial \psi} = \frac{\partial}{\partial \psi}\left[a(T - T_c)\psi^2 + \frac{\beta}{2}\psi^4\right] = 2a(T - T_c)\psi + 2\beta\psi^3$$

Extreme points are found at $\psi = 0$ and $\psi = \pm\sqrt{\frac{-a}{\beta}(T - T_c)}$.
For $T \geq T_c$, $\psi_0(T \geq T_c) = 0$ gives the minimum, i.e. $\mathcal{F}_0(T \geq T_c) = 0$.
For $T \leq T_c$, $\psi_0(T \leq T_c) = \pm\sqrt{\frac{-a}{\beta}(T - T_c)}$ is the minimum, giving free energy

$$\mathcal{F}_0(T \leq T_c) = \frac{-a^2}{\beta}(T - T_c)^2 + \frac{a^2}{2\beta}(T - T_c)^2 = \frac{-a^2}{2\beta}(T - T_c)^2 \leq \mathcal{F}_0(T \geq T_c).$$

For the specific heat, we find

$$C(T) = -T\frac{\partial^2 \mathcal{F}}{\partial T^2} = \begin{cases} 0 & T > T_c \\ \frac{a^2}{\beta}T & T < T_c \end{cases}.$$

There is thus a discontinuity in $C(T)$ at $T = T_c$ with size $\Delta C(T) = \frac{a^2}{\beta}T_c$. See the following sketches of the T-dependence of the derived quantities.

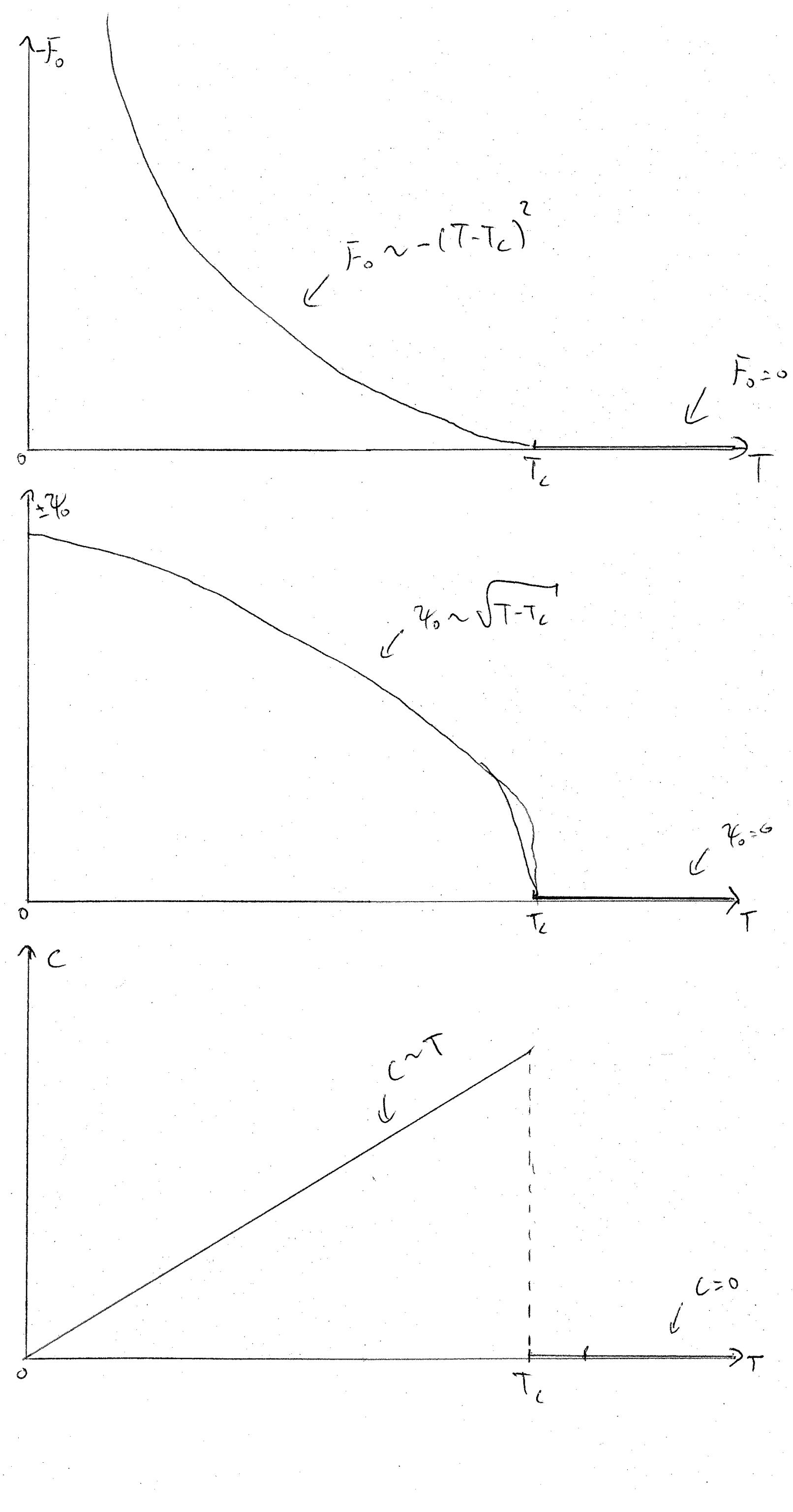

Figure 1: For the Landau theory, we find the drawn temperature dependences for equilibrium values of $\mathcal{F}_0$, ψ_0 and C. Note the minus sign for the free energy $\mathcal{F}_0$. At $T = 0$, there are y axis intersections for all three quantities, namely a minimum $\mathcal{F}_0(0) = \frac{-a^2}{2\beta}T_c^2$, $\psi_0(0) = \pm\sqrt{\frac{-a}{b}T_c}$, and $C(0) = 0$, which I forgot to indicate in the sketches. Do also note that there thus is an intersection in the $\mathcal{F}_0(T)$ curve at $T = 0$, although the drawing may look asymptotic.

5 Type-I superconducting foil

1. The screening equation is given as

$$\nabla^2 \vec{B} = \frac{\vec{B}}{\lambda^2}.$$

For easy of calculation, we will use cartesian coordinates, and put the external magnetic field B_E along the x axis: $\vec{B_E} = B_E \hat{x}$. A foil with thickness a we put parallel to the xy plane with the middle of the thickness at $z = 0$ such that the foil fills $-\frac{a}{2} < z < \frac{a}{2}$. Due to symmetry in the xy plane of the system, the field inside the foil can only depend on z coordinates. So we define the magnitude of the field $|\vec{B}| = B(z)$.

Using the screening equation, we look for a solution.

$$\nabla^2 \vec{B} = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] \vec{B}$$

we realize that we only have z dependence, and $B_y = 0 = B_z$.

$$\nabla^2 \vec{B} = \frac{\partial^2 B_x}{\partial z^2} \hat{x}$$

Rewriting yields

$$\vec{B} = \lambda^2 \frac{\partial^2 B_x}{\partial z^2} \hat{x}.$$

For this we know the general solution:

$$\vec{B} = B_0 \left[C \cdot e^{z/\lambda} + D \cdot e^{-z/\lambda}\right],$$

with constants B_0, C, and D.

Now we can apply two boundary conditions to find the solution inside the material.

First, due to mirror symmetry in z, we require $B(z) = B(-z)$, giving that $C = D$, thus we contract the constants as $B_0' = CB_0 = DB_0$. This allows us to write the exponents into *cosh* form.

$$B(z) = B_0' \left[e^{z/\lambda} + e^{-z/\lambda}\right] = B_0' \cosh \frac{z}{\lambda}$$

Second, just outside the foil, at $z = \pm\frac{a}{2}$, the field must be B_E, and the field should be continuous across the boundary:

$$B_E = B(\frac{a}{2}) = B_0' \cosh \frac{a}{2\lambda} \iff B_0' = \frac{B_E}{\cosh \frac{a}{2\lambda}}$$

This gives us our final expression for $B(z)$:

$$B(z) = \begin{cases} B_E \frac{1}{\cosh \frac{a}{2\lambda}} \cosh \frac{z}{\lambda} & |z| \leq \frac{a}{2} \\ B_E & |z| \geq \frac{a}{2} \end{cases}.$$

The supercurrent follows from the Maxwell-Ampère law, considering that there are no other currents, and we look at a current steady over time ($\frac{\partial \vec{E}}{\partial t} = 0$):

$$\nabla \times \vec{B}(z) = \mu_0 \vec{J_s}$$

Reordering and calculating the curl gives:

$$\vec{J_s} = \frac{1}{\mu_0} \nabla \times (\frac{\partial B(z)}{\partial z} \hat{x}) = \frac{B_E}{\mu_0 \lambda \cosh \frac{a}{2\lambda}} \sinh \frac{z}{\lambda} \hat{y}$$

2. From the derivation of the Ginzburg-Landau theory, we get the following expression for the supercurrent $\vec{J}_s$:
$$\vec{J}_s = -\frac{2e\hbar n_s}{m}(\nabla\theta + \frac{2e\vec{A}}{\hbar})$$
Using the rigid gauge, we set $\theta = 0$. Next, we can equate the previously found supercurrent for our foil to the Ginzburg-Landau found one and reorder to find $\vec{A}$:
$$\vec{A} = \frac{-B_E m \sinh\frac{z}{\lambda}}{4\lambda\mu_0 e^2 n_s \cosh\frac{a}{2\lambda}}\hat{y}$$
This we can rewrite using $\lambda = \sqrt{\frac{m}{\mu_0 n_s e^2}}$ for the London penetration depth as
$$\vec{A} = \frac{-B_E \lambda \sinh\frac{z}{\lambda}}{4\cosh\frac{a}{2\lambda}}\hat{y}.$$

3. $$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{\partial 0}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial 0}{\partial z} = 0,$$
as A_y is independent of y.

In our case, indeed the rigid gauge choice gives the criterium for the London gauge ($\nabla\cdot\vec{A} = 0$).

In the rigid gauge, the order parameter ψ is constant in space and time. To then also have that $\nabla\cdot\vec{A} = 0$, follows from the expression for the supercurrent as we saw earlier.

Reversely, assume that $\nabla\cdot\vec{A} = 0$, and look at what conditions need to be met in order to imply rigid gauge. Again, we look at the expression for the supercurrent as function of θ and $\vec{A}$,
$$\begin{aligned}\vec{J}_s &= -\frac{2e\hbar n_s}{m}(\nabla\theta + \frac{2e\vec{A}}{\hbar}) \\ \iff \frac{2e}{\hbar}\vec{A} &= -\frac{m}{2e\hbar n_s}\vec{J}_s - \nabla\theta,\end{aligned}$$
and take the divergence,
$$\frac{2e}{\hbar}\nabla\cdot\vec{A} = -\frac{m}{2e\hbar n_s}\nabla\cdot\vec{J}_s - \Delta\theta = 0 \implies \Delta\theta = -\frac{m}{2e\hbar n_s}\nabla\cdot\vec{J}_s.$$
This is what the London gauge implies. Now the question is under what circumstances the rigid gauge follows from the London gauge. This is the case for $\nabla\cdot\vec{J}_s = 0$, or in words, when there is conservation of superelectrons. If this is not the case (if the divergence is non-zero), there is conversion between normal electrons and superelectrons. This would take place if the temperature is lowered, as more superelectrons allow for a larger supercurrent, thus a larger critical magnetic field. This result seems to agree with Waldram's conclusion in [1, p. 24–26].

4. We apply a gauge transformation as follows.
$$\chi(\vec{r},t) = \frac{-\hbar}{2e}(\omega t - \vec{k}\cdot\vec{r}) \tag{1}$$
$$\vec{A} \to \vec{A} + \nabla\chi = \vec{A} + \frac{\hbar}{2e}\vec{k} \tag{2}$$
$$\phi \to \phi - \frac{\partial\chi}{\partial t} = \phi + \frac{\hbar}{2e}\omega \tag{3}$$

6 Type II superconductors and the vortex lattice

In 2003, Alexei Abrikosov was one of the winners of the Nobel Prize in Physics "for pioneering contributions to the theory of superconductors and superfluids". For this occasion, he gave a lecture called "Type II superconductors and the vortex lattice"[2] explaining the discoveries that led to the understanding of conventional superconductors. To get started, let me first explain what superconductors are.

Superconductors are characterized by perfect diamagnetism and zero resistance. Perfect diamagnetism is the ability by superconductors to have a net zero magnetic field inside. If you apply an external magnetic field, this thus means that a superconductor will let a current flow on its inside to generate a field to counteract this external field $\vec{H}$. This generated current is called a supercurrent. Superconductivity is, however, a phase of the material. Superconductors only have these properties below a certain temperature, its critical temperature T_c, and can only expel a maximum external magnetic field, its critical magnetic field $B_c(T)$, which is a function of the temperature. The zero resistance property follows from the perfect diagmagnetism. It is impossible for the material to let these supercurrents flow indefinitely with resistance, as heat would be generated.

The class of superconductors we have a model for, is the class of conventional superconductors. In this class, there are two types, called type-I and type-II superconductors.

In type-I superconductors, there is only one phase in which the superconductor material exhibits perfect diamagnetism: when the externally applied magnetic field $H < B_c(T)$ and $T < T_c$.

In type-II superconductors, there are two phases distinct from the normal conducting state. One is the superconducting state which behaves as in type-I superconductors, with critical field $B_{c1}(T)$. This state is reached for $T < T_c$ and $B_E < B_{c1}(T)$. The other state is a mixed state that allows some flux to pass through the material. This passing through is done by creating normally conducting channels throughout the material where a fixed amount of flux can pass through. This fixed amount is a multiple of the flux quantum Φ_0. The material generates current around these channels in accordance to the Maxwell-Ampère law, conforming to the let through magnetic field inside the vortex and cancelling the field on the outside the vortex.

There are lots of applications for both the perfect diagmagnetism and the zero resistivity. There is even a Wiki about them: `https://en.wikipedia.org/wiki/Technological_applications_of_superconductivity`. What is most notable about these applications, is that maintaining a temperature below the critical temperature is the biggest challenge. A real breakthrough for superconductivity would be the discovery of room-temperature superconductors at atmospheric pressure, or materials close to that. Currently, the highest T_c material we know is carbonaceous sulfur hydride (CH_8S) with $T_c = 15\,°C$ but at a pressure of a whopping 267 GPa. At atmospheric pressure, the highest T_c material known is a cuprate [3] $HgBa_2Ca_2Cu_3O_{8+\delta}$ at $T_c = 135\,K$. The quest for this breakthrough is intensely researched, although most is experimental. The clue is that most of the high T_c materials that are being discovered, are unconventional superconductors. As there is no theory for them (yet), the search is mostly educational guessing. By trying to find patterns in the previously high T_c materials, similar materials are studied to see if they also exhibit superconductivity. One of the patterns is that superconductivity in cuprates is high T_c. We'll focus on these materials in the following.

Currently, most hopeful candidates are cuprates. These materials are made of layers of copper oxides (CuO_2) alternated with layers of other metal oxides. The copper oxide layers are the superconductive layers, and the other metal oxides are used as charge reservoirs, doping electrons (or holes) into the copper oxide layers. Due to the geometry of these materials, there is anisotropy in the resistivity of the material. Parallel to the layers, superconductivity takes place in the copper oxide layers. Perpendicular to the layers, this is not the case.

The behaviour of the material can be tuned by tuning the doping, thus the other metal oxides as mentioned before. A typical phase diagram as function of the doping can be seen

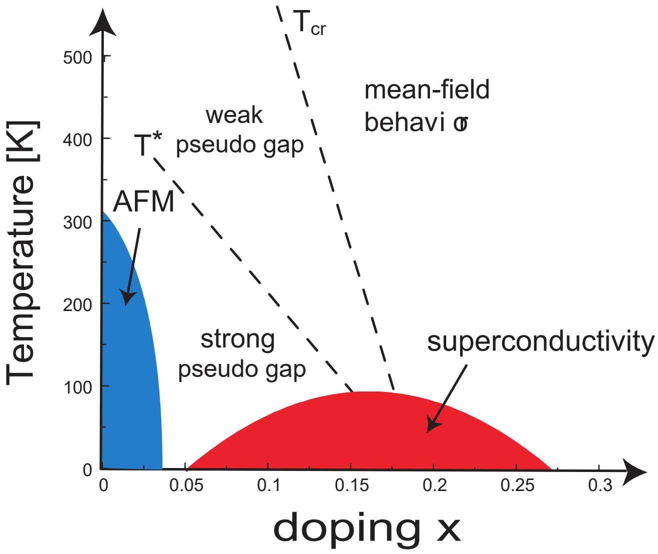

Figure 2: For high T_c superconducting cuprates, a typical phase diagram as function of doping looks like this.[4]

in figure 2. The material can be steered from being antiferrimagnetic to superconductive by increasing doping.

As can be seen, there is an optimal doping fraction for achieving highest T_c. Aiming for this doping yields the desired material.

Now the question is what direction to search for. The timeline in figure 3 might give a direction for the most promising types of cuprates to look into. It could be, however, that other types have higher T_c. A lot of creativity is therefore needed to find them.

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The nobel prize lecture by Abriskosov [2] was really interesting. The start was a good recap of the breakthroughs relevant to conventional superconductivity,[1] but in pages 61–63, the theory is worked through a little quickly. I might reread it some times.

7 Currents inside type-II superconducting cylinder

For $B_{c1} < B_E < B_{c2}$, the cylinder of type-II superconductor material is in the mixed state. In the mixed or vortex state, superconductors let through a number of finite flux quanta Φ_0. Some small regions of the material are not superconducting, but in the normal state. Flux passes through these regions in multiples of Φ_0, but usually just one Φ_0 per region, and a supercurrent is generated to expel the field from the rest of the material. These flux allowing regions are called vortices, due to their shape and direction of current flow. Vortices look like channels (or tubes), and supercurrents move around these channels in a spiraling fashion. One can visualize this as current through a coil such that on the inside of the coil, the field is in one direction perpendicular to it, and on the outside it is the opposite direction. The current direction is governed by the Maxwell-Ampère equation. In this case, the current is such that the field inside the cylinder but outside these channels is counteracted.

Please see the figure below for a beautiful drawing. It was not specified what the direction of $\vec{B_E}$ was with respect to the cylinder orientation, so I chose what I thought was most reasonable as an example.

[1]Why is it that every story on superconductivity includes KGB captivity?

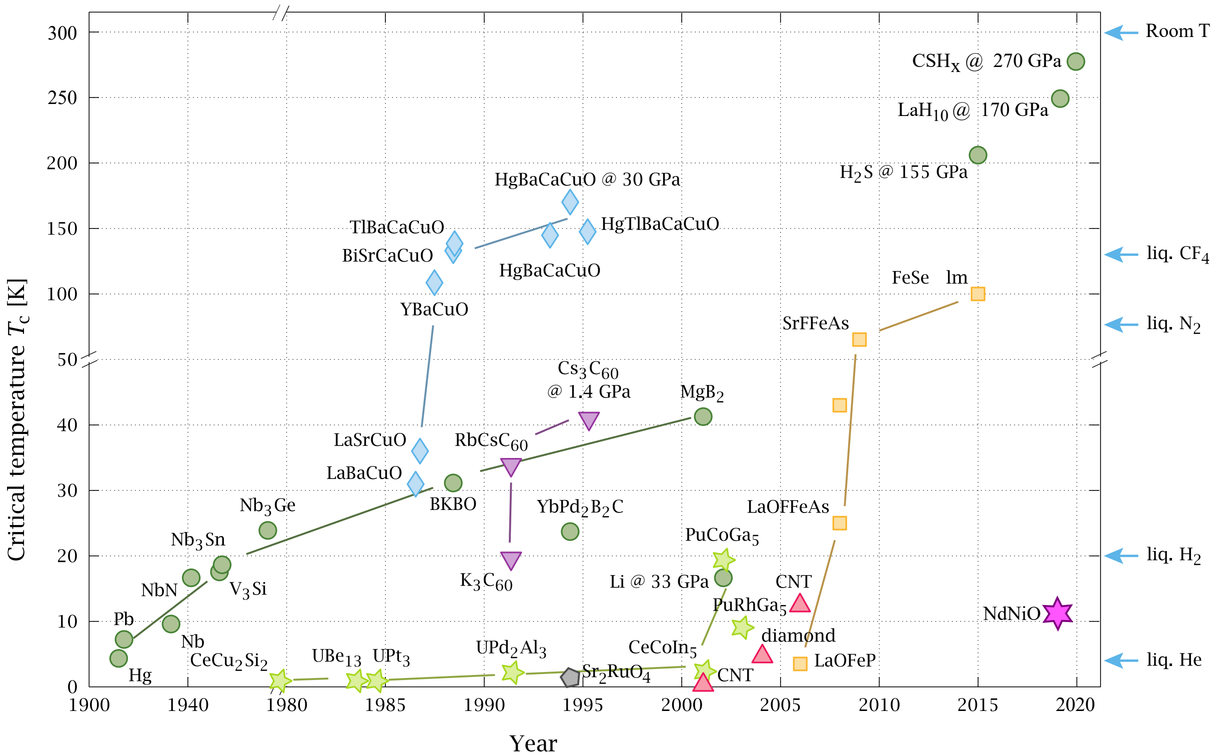

Figure 3: The last century, a lot of research has been done in the direction of cuprate superconductivity. Pia Jensen Ray made this overview for his master thesis.[5] The different paths are different types of cuprates. Please see his thesis for the meaning of the labels. On the right side, an idea of the temperature is givin by comparing it to common cooling agents.

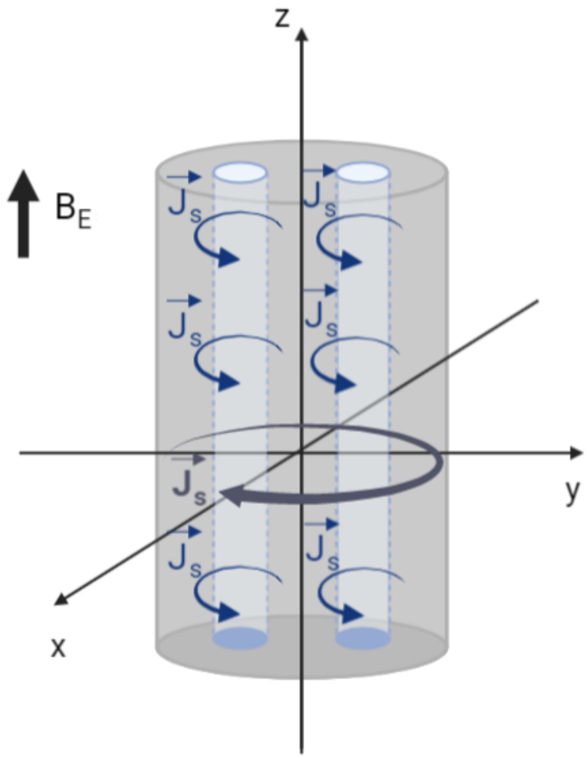

Figure 4: The direction of $\vec{J}_s$ is such that a magnetic field is generated to counteract and even expel the external field outside the vortices inside the material. Around the vortices, that means that the supercurrents run anti-clockwise. The field is then along $\vec{B_E}$ inside the vortices, but along $-\vec{B_E}$ outside the vortices but inside the material. Around the outside border of the cylinder, however, $\vec{J}_s$ runs clockwise and again cancels $\vec{B_E}$ on the inside of the material.

References

[1] Waldram JR. Superconductivity of metals and cuprates. Bristol ; Philadelphia, Pa: Institute of Physics Pub; 1996.

[2] Abrikosov AA. Type II superconductors and the vortex lattice; 2003. Available from: https://www.nobelprize.org/prizes/physics/2003/abrikosov/lecture/.

[3] Dai P, Chakoumakos BC, Sun GF, Wong KW, Xin Y, Lu DF. Synthesis and neutron powder diffraction study of the superconductor HgBa2Ca2Cu3O8 + by Tl substitution. Physica C: Superconductivity. 1995 Mar;243(3):201–206. Available from: https://www.sciencedirect.com/science/article/pii/0921453494024618.

[4] Chubukov A, Pines D, Schmalian J. A Spin Fluctuation Model for D-wave Superconductivity. 2002 02;p. 51.

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