Superconductivity - Assignment 1

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1 Electron-phonon coupling in elements

Conventional superconductors are described by considering Cooper pairs: pairs of electrons mediated by electron-phonon coupling. This is usually described by BCS theory. The hypothesis is that stronger electron-phonon coupling results in enhanced critical temperatures for the superconducting phase transition. In order to investigate this, we need a way to determine the electron-phonon coupling strength. We will attempt to do this by looking at the room temperature resistivity of superconducting elements.

For metals, we have the following familiar relation for resistivity ρ over temperature T.

$$\rho = \underbrace{\rho_0}_{\text{impurities}} + \underbrace{aT^2}_{\text{electron-electron coupling}} + \underbrace{bT^5}_{\text{electron-phonon coupling}} \tag{1}$$

At $T = 0$, only resistivity due to impurities and lattice defects is left in the material. Then, at low temperatures, electron-electron coupling increases resistivity. The effect that is the largest at room temperature, is due to electron-phonon interaction, due to the fifth power in temperature. The constants a and b differ from material to material. If the hypothesis is correct, an increasing trend of critical temperature T_c over room temperature resistivity ρ_{300K} should be observed.

For a collection of superconducting elements, this relation is plotted in figure 1. The data on critical temperatures T_c and (approximately) room temperature resistivity $\rho_{300\,\mathrm{K}}$ is from various sources, as can be found in the table in appendix A.

Looking at the plot, there is no obvious positive trend between T_c and ρ_{300K}. As a way to quantize this (lack of) correlation, we can take a look at the Pearson correlation coefficient: $r = 0.165415$. Pearson's r is a measure of linear correlation. If $|r| = 1$, there is a perfectly linear relation. The lower $|r|$ is, the less correlated the points are. The sign of r gives the direction of the trend. This slightly positive value found for the superconducting elements suggests a slightly positive but uncertain correlation. As the relation between electron-phonon coupling and resistivity is well-established, it seems reasonable to conclude that there is no unambiguous relation between T_c and ρ_{300K}. There seem to be other factors we are missing in this analysis.

Looking at the plot, however, it would be too easy to conclude there is no relation between these quantities at all. There do seem to be two branches with approximate linear correlation. Distinguishing these two groups roughly along the line from the origin under Re, we find $r = 0.69739$ and $r = 0.621341$ respectively above and below the line.

Another missing factor is the comparison to non-superconductor elemental metals. Although they do not really experience a phase transition to a superconducting state, so they do not have a finite T_c associated to them, they could be plotted having $T_c = 0$. I chose to exclude them from the plot, as they would only clutter it further, and there is no real relation visible.

Further distinction could be in superconductor type. Most of the plotted elements are type-I superconductors, but vanadium, for example, is type-II. Vanadium does, however, behave similar to the rest of the elements.

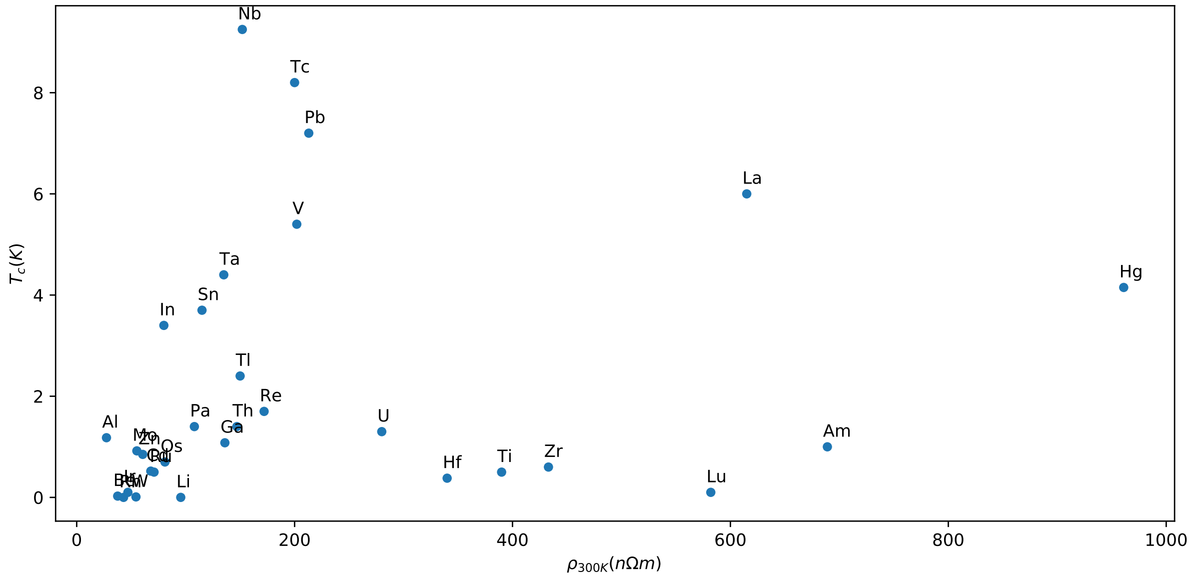


Figure 1: In this plot of the critical temperature T_c versus the room temperature resistivity ρ_{300K} for elemental superconductors, not one clear relation can be distinguished. For most elements, resistivity is taken at room temperature $T = 300\,\mathrm{K}$. If it was unavailable in consulted references, the value at the temperature closest to 300 K was chosen. See the table in appendix A for the raw data including their source. The mess in the left bottom corner was hard to filter out. A log-log plot was attempted and improved separation, but obscured the observed two branches in this linear plot.

In conclusion, the statement that superconductivity is more likely to be observed in metals whose normal state is highly resistive is rejected by this analysis of the superconducting elements.

2 Exam question electrodynamics in superconductors

In 1935, the London brothers established a first theory to explain the Meissner-Ochsenfeld effect. Although their theory was not completely correct, they derived two correct and important electrodynamical equations, which we know as the London equations. The first and second London equations are

$$\frac{\partial}{\partial t}(\Lambda \vec{J}_s) = \vec{E} \qquad -\nabla \times (\Lambda \vec{J}_s) = \vec{B},$$

with $\Lambda = \frac{m_e}{n_s e^2}$, m_e the normal electron mass, n_s the superfluid electron density.

1. Describe the idea behind the London model for the Meissner-Ochsenfeld effect.

   The London model is based on the two fluid model that was proposed by Lev Landau a few years prior to the London model. The idea of the London model is the coexistence of two types of electrons in a superconductor, namely normal electrons and superelectrons. Both types can transfer electrical current, the normal electrons with resistance, the super-electrons without, and they do so in parallel. Heat can only be transferred by the normal electrons.

   Electrons can change type. At high temperature, all the electrons are of the normal type. Below a critical temperature T_c, however, the density of superelectrons quickly emerges and a supercurrent can flow.

2. Using the second London equation, derive the screening equation

$$\nabla^2 \vec{B} = \frac{\vec{B}}{\lambda_L^2}$$

with $\lambda_L^2 = \frac{\Lambda}{\mu_0}$.
Use the Maxwell-Ampère law

$$\nabla \times \vec{B} = \mu_0(\vec{J} + \frac{\partial \vec{E}}{\partial t})$$

We assume a steady-state, so starting from the Maxwell-Ampère law, we can immediately cross off the derivative $\frac{\partial \vec{E}}{\partial t} = 0$. *As we are in the superconducting state, the total current is mostly due to current of superelectrons (supercurrent), as these flow without resistance and thus flow way easier. So we set* $\vec{J} = \vec{J}_s$. *We now have*

$$\nabla \times \vec{B} = \mu_0(\vec{J}_s).$$

Taking the curl of both sides, we end up with

$$\nabla \times (\nabla \times \vec{B}) = \nabla^2 \vec{B} = \mu_0 \nabla \times \vec{J}_s.$$

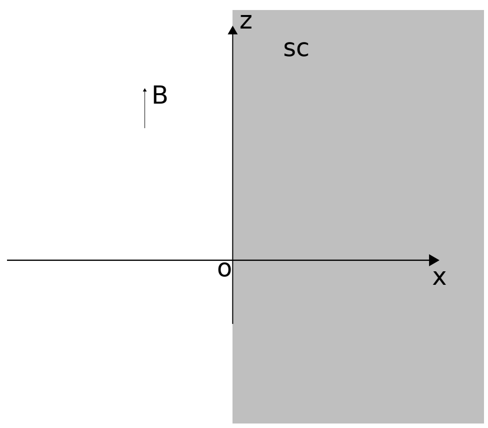
Assuming that Λ *is constant over the material, we have from the second London equation that*

$$\nabla \times (\Lambda \vec{J}_s) = -\Lambda \nabla \times \vec{J}_s = \vec{B},$$

which we can easily substitute into our early result. This way we find our result:

$$\nabla^2 \vec{B} = \frac{\mu_0 \vec{B}}{\Lambda} = \frac{\vec{B}}{\lambda_L^2}$$

3. Assume we have the situation as sketched in figure 3. A superconducting slab is placed at $x = 0$ extending to infinity in both x, $\pm y$ and $\pm z$ directions. A uniform external magnetic field $\vec{B} = B\hat{x}$ is applied. Use the just derived screening equation to calculate the field $\vec{B}$ inside the superconductor.



Due to symmetry in both z *and* y *directions, we can already tell that the result will depend on* x *only. So for a* $\vec{B} = \vec{B}(x)$,

$$\nabla^2 \vec{B} = \frac{\partial^2 \vec{B}}{\partial x^2},$$

Now from the screening equation we find

$$\nabla^2 \vec{B} = \frac{\partial^2 \vec{B}}{\partial x^2} = \frac{\vec{B}}{\lambda_L^2}.$$

The familiar solution to this is the superposition of exponents to the left and right:

$$\vec{B}(x) = [c_1 e^{x/\lambda_L} + c_2 e^{x/\lambda_L}]\hat{z}.$$

Using that $\vec{B}(0) = \vec{B}$, and that the field cannot diverge for large x, we find that $c_1 = 0$ and $c_2 = B$, so that

$$\vec{B}(x) = \vec{B} e^{-x/\lambda_L}.$$

3 Difference between type-I and type-II superconductors

In the realm of conventional superconductors, we have type-I and type-II superconductors. Both types are mediated by electron-phonon coupling, but there are quite some differences. Using figure 3, we will go through their differences. The material covered is mostly from the Solid State Physics course by Steffen Wiedmann and the Superconductivity course by Alix McCollam, with some statistical physics knowledge due to Mikhail Katsnelson's Advanced Statistical Physics. Holes in my knowledge were mostly filled by [1].

Type-I superconductors (sc) are described by BCS theory. An example of a type-I sc is lead (Pb). Below a critical temperature T_c, these materials exhibit perfect diamagnetism. Inside the sc, a magnetic field is generated to expell the externally applied field, such that the net field is zero. This is called the Meissner-Ochsenfeld effect. The perfect diamagnetism is quantized as having susceptibility $\chi = -1$. There is, however, a limit to how large a field can be completely expelled. This is called the critical field H_c. If H_c is exceeded, the superconductivity breaks down, thus the material will cease to expell the field, the diagmagnetism drops to $\chi = 0$. This effect is seen in the bottom-left plot in figure 3. The relation between these two critical values is given as

$$H_c(T) = H_c(0) \left[1 - \frac{T}{T_c}^2 \right].$$

An example of this curve is plotted in the top-left of figure 3. Two phases can be distinguished: the Meissner (or superconducting) state under the graph, and the normal state outside it. The transition between these states is a first-order phase transition due to the discontinuity in the magnetization $M = \frac{dF}{dH}$, the first derivative of the free energy to the applied field.

Then there are type-II sc, for example, niobium (Nb). Instead of only having a Meissner or sc phase, they have another phase: the vortex state. Below T_c and lower critical field $H_{c1}(T)$, the type-II sc is in the Meissner state, and the material thus completely cancels the externally applied field. Above the upper critical field $H_{c2}(T)$, the material is in the normal state. When the field is between these two critical fields, $H_{c1}(T) < H < H_{c2}(T)$, the material is in the vortex state. See the top-right plot in figure 3. In the vortex state, the externally applied magnetic field is not completely expelled. Instead, the material consists of normal and sc regions, the former called vortices. These vortices are normal conducting regions and allow magnetic flux to pass. The flux let through by the vortices is quantized by flux quanta, and more flux is let through by allowing more quanta through the material. In practice, this means that more vortices appear, which is energetically more favorable than having vortices with more flux. As can be seen in the magnetization graph on the bottom-right of figure 3, the magnetization is continuous over the temperature, thus no first-order phase transition is observed. A second-order phase transition, however, does take place at H_{c1}, and is due to a discontinuity in $\chi = \frac{d^2F}{dH^2}$. The transition from the vortex to the normal state is also of second order, as the magnetization graph is continuous but the magnetization (order paramater) disappears above H_{c2}.

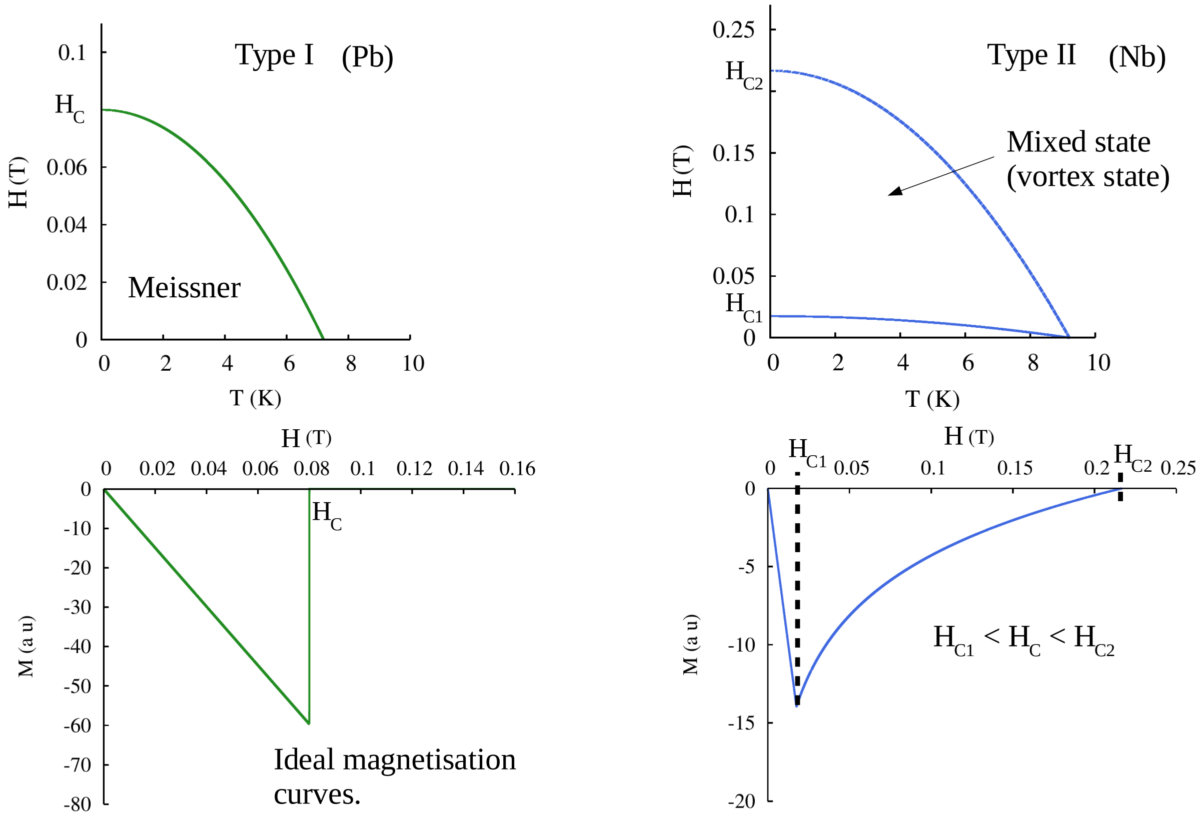


Figure 2: On the left, the behaviour of a type-I superconductor is displayed, on the right the type-II. The top graphs show the critical fields H_c over the temperature of the sc. The bottom graphs show the magnetization of the materials over the temperature. Phase transitions for both types are clearly visible. The figure is borrowed from the presentation of week two of the Superconductivity course by Alix McCollam in 2022.

References

[1] Waldram JR. Superconductivity of metals and cuprates. Bristol ; Philadelphia, Pa: Institute of Physics Pub; 1996.

[2] Webb GW, Marsiglio F, Hirsch JE. Superconductivity in the elements, alloys and simple compounds. Physica C: Superconductivity and its Applications. 2015 Jul;514:17–27. Available from: `https://linkinghub.elsevier.com/retrieve/pii/S0921453415000647`.

[3] Chemical Rubber Company. CRC handbook of chemistry and physics: a ready-reference book of chemical and physical data. 84th ed. Lide DR, editor. Boca Raton: CRC Press; 2003.

[4] Electrical resistivity and conductivity; 2022. Page Version ID: 1070466789. Available from: `https://en.wikipedia.org/w/index.php?title=Electrical_resistivity_and_conductivity&oldid=1070466789`.

[5] Šmidt VV, Müller P. The Physics of superconductors: introduction to fundamentals and applications. Berlin: Springer; 2010.

A Superconducting elements

Element	T_c(K)	ρ_{300K}(nΩ m)	Source T_c	Source ρ_{300K}
Nb	9.2500	152.00	[2]	at 273 K [3]
Tc	8.2000	200.00	[2]	[4]
Pb	7.2000	213.00	[2]	[3]
La	6.0000	615.00	[2]	[3]
V	5.4000	202.00	[2]	[3]
Ta	4.4000	135.00	[2]	[3]
Hg	4.1500	961.00	[2]	at 298 K [3]
Sn	3.7000	115.00	[2]	at 273 K [3]
In	3.4000	80.00	[2]	at 273 K [3]
Tl	2.4000	150.00	[2]	at 273 K [3]
Re	1.7000	172.00	[2]	at 273 K [3]
Th	1.4000	147.00	[2]	at 273 K [3]
Pa	1.4000	108.00	[2]	[3]
U	1.3000	280.00	[2]	at 273 K [3]
Al	1.1800	27.33	[2]	[3]
Ga	1.0800	136.00	[2]	at 273 K [3]
Am	1.0000	689.00	[2]	[4]
Mo	0.9200	55.20	[2]	[3]
Zn	0.8500	60.60	[2]	[3]
Os	0.7000	81.00	[2]	at 273 K [3]
Zr	0.6000	433.00	[2]	[3]
Cd	0.5200	68.00	[2]	at 273 K [3]
Ru	0.5000	71.00	[2]	at 273 K [3]
Ti	0.5000	390.00	[2]	at 273 K [3]
Hf	0.3800	340.00	[2]	[3]
Ir	0.1000	47.00	[2]	at 273 K [3]
Lu	0.1000	582.00	[2]	[3]
Be	0.0260	37.60	[5] (but [2] reports 1440 K)	[3]
W	0.0100	54.40	[2]	[3]
Li	0.0004	95.50	[2]	[3]
Rh	0.0003	43.00	[2]	at 273 K [3]