Superconductivity - Assignment 2

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4 Temperature dependence in Landau model

In the Landau model, free energy is given as function of order parameter ψ and temperature T as

$$\mathcal{F} = a(T - T_c)\psi^2 + \frac{\beta}{2}\psi^4.$$

The equilibrium state as function of temperature T is the state of minimal free energy with respect to the order parameter $\psi(T)$. This point we call $F_0(T)$ with order parameter $\psi_0(T)$. For this, we will take the derivative of F with respect to ψ and equate it to zero.

$$0 = \frac{\partial \mathcal{F}}{\partial \psi} = \frac{\partial}{\partial \psi}\left[a(T - T_c)\psi^2\right] = 2a(T - T_c)\psi + 2\beta\psi^3$$

Extreme points are found at $\psi = 0$ and $\psi = \pm\sqrt{\frac{-a}{\beta}(T - T_c)}$.
For $T \geq T_c$, $\psi_0(T \geq T_c) = 0$ gives the minimum, i.e. $\mathcal{F}_0(T \geq T_c) = 0$.
For $T \leq T_c$, $\psi_0(T \leq T_c) = \sqrt{\frac{-a}{\beta}(T - T_c)}$ is the minimum, giving free energy

$$\mathcal{F}_0(T \leq T_c) = \frac{-a^2}{\beta}(T - T_c)^2 + \frac{a^2}{2\beta}(T - T_c)^2 = \frac{-a^2}{2\beta}(T - T_c)^2 \leq \mathcal{F}_0(T \geq T_c)$$

where we chose the positive of the $\pm$ as the order parameter is understood to increase from finite at the phase transition.

Is this a reasonable statement? It actually does not really matter that much as mostly ψ^2 is used, but the physical meaning is totally different. It implies some kind of symmetry, too. It seems that also [1] mentions this.

For the specific heat, we find

$$C(T) = -T\frac{\partial^2 \mathcal{F}}{\partial T^2} = \begin{cases} 0 & T > T_c \\ \frac{a^2}{\beta}T & T < T_c \end{cases}.$$

There is thus a discontinuity in $C(T)$ at $T = T_c$ with size $\Delta C(T) = \frac{a^2}{\beta}T_c$.

5 Type-I superconducting foil

1. The screening equation is given as

$$\nabla^2 \vec{B} = \frac{\vec{B}}{\lambda}.$$

For easy of calculation, we will use cartesian coordinates, and put the external magnetic field B_E along the x axis: $\vec{B_E} = B_E\hat{x}$. A foil with thickness a we put parallel to the xy plane with the middle of the thickness at $z = 0$ such that the foil fills $-\frac{a}{2} < z < \frac{a}{2}$. Due to symmetry in the xy plane of the system, the field inside the foil can only depend on z coordinates. So we define the magnitude of the field $|\vec{B}| = B(z)$.

Using the screening equation, we look for a solution.

$$\nabla^2 \vec{B} = \nabla(\nabla \cdot \vec{B}) - \nabla \times (\nabla \times \vec{B})$$

$\vec{B}$ is divergenceless, so we are left with the latter term. Next, we take the curls writing B_i for the ith component of $\vec{B}$, and realize that we only have z dependence, and $B_y = 0 = B_z$.

$$-\nabla \times (\nabla \times \vec{B}) = -\nabla \times (\frac{\partial B_x}{\partial z}\hat{y}) = -(-\frac{\partial^2 B_x}{\partial z^2}\hat{x})$$

Rewriting yields

$$\vec{B} = \lambda \frac{\partial^2 B_x}{\partial z^2}\hat{x}.$$

For this we know the general solution:

$$\vec{B} = B_0 \left[C \cdot e^{z/\lambda} + D \cdot e^{-z/\lambda} \right],$$

with constants B_0, C, and D.

Now we can apply two boundary conditions to find the solution inside the material.

First, due to mirror symmetry in z, we require $B(z) = B(-z)$, giving that $C = D$, thus we contract the constants as $B_0' = CB_0 = DB_0$. This allows us to write the exponents into *cosh* form.

$$B(z) = B_0' \left[e^{z/\lambda} + e^{-z/\lambda} \right] = B_0' \cosh \frac{z}{\lambda}$$

Second, just outside the foil, at $z = \pm \frac{a}{2}$, the field must be B_E, and the field should be continuous across the boundary:

$$B_E = B(\frac{a}{2}) = B_0' \cosh \frac{a}{2\lambda} \iff B_0' = \frac{B_E}{\cosh \frac{a}{2\lambda}}$$

This gives us our final expression for $B(z)$:

$$B(z) = \begin{cases} B_E \frac{1}{\cosh \frac{a}{2\lambda}} \cosh \frac{z}{\lambda} & |z| \leq \frac{a}{2} \\ B_E & |z| \geq \frac{a}{2} \end{cases}.$$

The supercurrent follows from the Maxwell-Ampère law, considering that there are no other currents, and we look at a current steady over time ($\frac{\partial \vec{E}}{\partial t} = 0$):

$$\nabla \times \vec{B}(z) = \mu_0 \vec{J}_s$$

Reordering and calculating the curl gives:

$$\vec{J}_s = \frac{1}{\mu_0} \nabla \times (\frac{\partial B(z)}{\partial z}\hat{x}) = \frac{B_E}{\mu_0 \lambda \cosh \frac{a}{2\lambda}} \sinh \frac{z}{\lambda}\hat{y}$$

2. From the derivation of the Ginzburg-Landau theory, we get the following expression for the supercurrent $\vec{J}_s$:

$$\vec{J}_s = -\frac{2e\hbar n_s}{m}(\nabla\theta + \frac{2e\vec{A}}{\hbar})$$

Using the rigid gauge, we set $\theta = 0$. Next, we can equate the previously found supercurrent for our foil to the Ginzburg-Landau found one and reorder to find $\vec{A}$:

$$\vec{A} = \frac{-B_E m \sinh \frac{z}{\lambda}}{4\lambda\mu_0 e^2 n_s \cosh \frac{a}{2\lambda}}\hat{y}$$

3.
$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{\partial 0}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial 0}{\partial z} = 0,$$

as $A_y \perp \hat{y}$, giving zero partial derivative.

In our case, indeed the rigid gauge choice gives the criterium for the London gauge ($\nabla \cdot \vec{A} = 0$).

In the rigid gauge, the order parameter ψ is constant in space and time. To then also have that $\nabla \cdot \vec{A} = 0$, follows from the expression for the supercurrent as we saw earlier.

Reversely, assume that $\nabla \cdot \vec{A} = 0$, and look at what conditions need to be met in order to imply rigid gauge. Again, we look at the expression for the supercurrent as function of θ and $\vec{A}$,

$$\begin{aligned} \vec{J}_s &= -\frac{2e\hbar n_s}{m}(\nabla\theta + \frac{2e\vec{A}}{\hbar}) \\ \iff \frac{2e}{\hbar}\vec{A} &= -\frac{m}{2e\hbar n_s}\vec{J}_s - \nabla\theta, \end{aligned}$$

and take the divergence,

$$\frac{2e}{\hbar}\nabla \cdot \vec{A} = -\frac{m}{2e\hbar n_s}\nabla \cdot \vec{J}_s - \Delta\theta = 0 \implies \Delta\theta = -\frac{m}{2e\hbar n_s}\nabla \cdot \vec{J}_s.$$

This is what only the London gauge implies. But when is then the rigid gauge applied by this? This is the case for $\nabla \cdot \vec{J}_s$, or, in words, when there is no conservation of supercurrent. If this is not the case (if the divergence is non-zero), there is conversion between normal current and supercurrent. This result seems to Waldram's conclusion in [2, p. 24–26].

4. We apply a gauge transformation as follows.

$$\chi(\vec{r}, t) = \frac{-\hbar}{2e}(\omega t - \vec{k} \cdot \vec{r}) \tag{1}$$

$$\vec{A} \to \vec{A} + \nabla\chi = \vec{A} + \frac{\hbar}{2e}\vec{k} \tag{2}$$

$$\phi \to \phi - \frac{\partial \chi}{\partial t} = \phi + \frac{\hbar}{2e}\omega \tag{3}$$

Do I really need to put in the previously found $\vec{A}$?

6 Type II superconductors and the vortex lattice

In 2003, Alexei Abrikosov was one of the winners of the Nobel Prize in Physics "for pioneering contributions to the theory of superconductors and superfluids". For this occasion, he gave a lecture called "Type II superconductors and the vortex lattice"[1] explaining the discoveries that led to the understanding of conventional superconductors. To get started, let me first explain what superconductors are.

Superconductors are characterized by perfect diamagnetism and zero resistance. Perfect diamagnetism is the ability by superconductors to have a net zero magnetic field inside. If you apply an external magnetic field, this thus means that a superconductor will let a current flow on its inside to generate a field to counteract this external field $\vec{H}$. This generated current is called a supercurrent. This is, however, a phase of the material. Superconductors only have these properties below a certain temperature, its critical temperature T_c, and can only expel a maximum external magnetic field, its critical magnetic field $B_c(T)$, which is a function of the temperature.

The class of superconductors we have a model for, is the class of conventional superconductors, which are explained by a theory called BCS (and some extensions). In this class, there are two types, called type-I and type-II superconductors.

In type-I superconductors, there is only one phase in which the superconductor material exhibits perfect diamagnetism: when the externally applied magnetic field $H < B_c(T)$ and $T < T_c$.

In type-II superconductors, there are two phases distinct from the normal conducting state. One is the superconducting state which behaves as in type-I superconductors, with critical field $B_{c1}(T)$. This state is reached for $T < T_c$ and $B_E < B_{c1}(T)$. The other state is a mixed state that allows some flux to pass through the material. This passing through is done by creating normally conducting channels throughout the material where a fixed amount of flux can pass through. This fixed amount is a multiple of the flux quantum Φ_0. The material generates current around these channels cancelling the field on the inside of the superconducting part of the material.

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The nobel prize lecture by Abriskosov [1] was really interesting. The start was a good recap of the breakthroughs relevant to conventional superconductivity,[1] but in pages 61–63, the theory is worked through a little quickly. I might reread it some times.

The essay so far is just a draft. Choosing a topic was hard. As we are to aim at bachelor students not knowing sc, I thought a proper introduction was appropriate.

7 Currents inside type-II superconducting cylinder

For $B_{c1} < B_E < B_{c2}$, the cylinder of type-II superconductor material is in the mixed state. In the mixed or vortex state, superconductors let through a number of finite flux quanta Φ_0. Some small regions of the material are not superconducting, but in the normal state. Flux passes through these regions in multiples of Φ_0, but usually just one Φ_0 per region, and a supercurrent is generated to expel the field from the rest of the material. This supercurrent moves around these region in a vortex-like shape.

Please see the figure below for a beautiful drawing. It was not specified what the direction of $\vec{B_E}$ was with respect to the cylinder orientation, so I chose what I thought was most reasonable as an example.

References

[1] Abrikosov AA. Type II superconductors and the vortex lattice; 2003. Available from: `https://www.nobelprize.org/prizes/physics/2003/abrikosov/lecture/`.

[2] Waldram JR. Superconductivity of metals and cuprates. Bristol ; Philadelphia, Pa: Institute of Physics Pub; 1996.

[1] Why is it that every story on superconductivity includes KGB captivity?

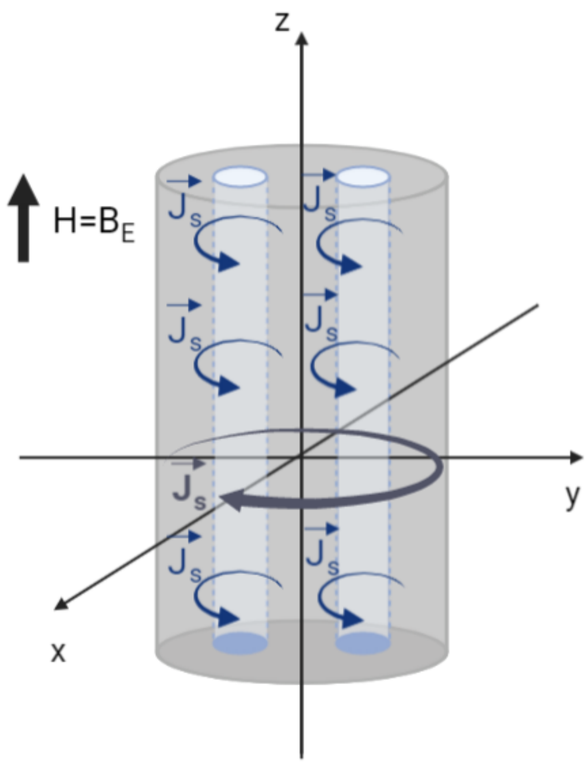

Figure 1: The direction of $\vec{J_s}$ is such that a magnetic field is generated to counteract and even expel the external field. Around the vortices, that menas that the supercurrents run anti-clockwise. Around the outside border of the cylinder, however, $\vec{J_s}$ runs clockwise to cancel $\vec{B_E}$.